Invariance or Covariance of Maxwell's field equations under Lorentz transformation:

Maxwell's field equation (Four equations) are

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\varepsilon_0} \quad \text{——— (i)} \\ \vec{\nabla}\cdot\vec{B} &= 0 \quad \text{——— (ii)} \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \quad \text{——— (iii)} \\ \vec{\nabla}\times\vec{B} &= \mu_0\left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \quad \text{——— (iv)} \end{aligned}\right\} \text{——— (1)}$$

These Maxwell's field equations can be written in terms of electromagnetic potential $\vec{A}$ and ϕ in two equations only as

$$\Box^2 \vec{A} = -\mu_0 \vec{J} \quad \text{——— (2)}$$

$$\Box^2 \phi = -\frac{\rho}{\varepsilon_0} \quad \text{——— (3)}$$

with the Lorentz condition

$$\operatorname{div}\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \quad \text{——— (4)}$$

The electromagnetic four potential (Four vector potential) A_μ and current four vector J_μ are defined as

$$A_\mu = \left(\vec{A}, \frac{i\phi}{c}\right) \quad \text{——— (5)}$$

$$\text{and} \quad J_\mu = (\vec{J}, ic\rho) \quad \text{——— (6)}$$

These four vectors obey the Lorentz transformation.

so $$A'_\mu = \alpha_{\mu\nu}\cdot A_\nu \quad \text{——— (7)}$$

and $$J'_\mu = \alpha_{\mu\nu}\cdot J_\nu \quad \text{——— (8)}$$

where
$$\alpha_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

Lorentz transformation of four vector potential A_μ is

$$\left.\begin{aligned} A_1' &= \gamma\left(A_1 - \frac{v}{c^2}\phi\right), \quad A_2' = A_2, \quad A_3' = A_3 \\ \text{and} \quad A_4' &= \gamma(A_4 - i\beta A_1) \quad \text{or} \quad \phi' = \gamma(\phi - vA_1) \end{aligned}\right\} \quad \text{———} \ (9)$$

Lorentz transformation of current four vector J_μ is

$$\left.\begin{aligned} J_1' &= \gamma(J_1 - v\rho), \quad J_2' = J_2, \quad J_3' = J_3 \\ \text{and} \quad J_4' &= \gamma(J_4 - i\beta J_1) \quad \text{or} \quad \rho' = \gamma\left(\rho - \frac{v}{c^2}J_1\right) \end{aligned}\right\} \quad \text{———} \ (10)$$

In terms of four vector potential A_μ and current four vector J_μ, Maxwell's field equation can be written in single equation as

$$\Box^2 A_\mu = -\mu_0 J_\mu \quad \text{———} \ (11)$$

with Lorentz condition $\frac{\partial A_\mu}{\partial x_\mu} = 0$ ——— (12)

The invariance (or covariance) of Maxwell's field equations require that in any inertial frame S', equations (11) and (12) must retain the same form, i.e, Maxwell's field equation will be invariant if

$$\left.\begin{aligned} \Box'^2 A_\mu' &= -\mu_0 J_\mu' \\ \text{with Lorentz condition} \quad \frac{\partial A_\mu'}{\partial x_\mu'} &= 0 \end{aligned}\right\} \quad \text{———} \ (13)$$

Now (a) $\Box'^2 A_1' = \Box^2 A_1'$ $\quad \because \Box'^2 = \Box^2$ D'Alembertian operator is invariant.

$$= \Box^2\left[\gamma\left(A_1 - \frac{v}{c^2}\phi\right)\right] \quad \text{using equ (9)}$$

$$= \gamma\left[\Box^2 A_1 - \frac{v}{c^2}\Box^2\phi\right]$$

$$= \gamma\left[-\mu_0 J_1 - \frac{v}{c^2}\left(-\frac{\rho}{\varepsilon_0}\right)\right] \quad \because \Box^2 A_1 = -\mu_0 J_1, \ \Box^2\phi = -\rho/\varepsilon_0$$

$$= \gamma(-\mu_0 J_1 + \mu_0 v\rho) \quad \because c^2 = \frac{1}{\mu_0\varepsilon_0} \Rightarrow \mu_0 = \frac{1}{c^2\varepsilon_0}$$

$$= -\mu_0[\gamma(J_1 - v\rho)]$$

$$\Rightarrow \Box'^2 A_1' = -\mu_0 J_1' \quad \text{———} \ 14(a) \quad \text{using equ (10)}$$

(b) $\Box'^2 A_2' = \Box^2 A_2' \quad \because \Box'^2 = \Box^2$

$= \Box^2 A_2 \quad \because A_2' = A_2$ using equ ⑨

$= -\mu_0 J_2$ using equ ⑬ $\Box^2 A_2 = -\mu_0 J_2$

$$\Box'^2 A_2' = -\mu_0 J_2' \text{ ——— } 14(b) \quad \because J_2' = J_2 \text{ using equ (10)}$$

(c) $\Box'^2 A_3' = \Box^2 A_3' \quad \because \Box'^2 = \Box^2$

$= \Box^2 A_3 \quad \because A_3' = A_3$ using equ ⑨

$= -\mu_0 J_3 \quad \because \Box^2 A_3 = -\mu_0 J_3$ using equ ⑪

$$\Box'^2 A_3' = -\mu_0 J_3' \text{ ——— } 14(c) \quad \because J_3' = J_3 \text{ using equ (10)}$$

(d) $\Box'^2 A_4' = \Box^2 A_4' \quad \because \Box'^2 = \Box^2$

$= \Box^2 [\gamma(A_4 - i\beta A_1)] \quad \because A_4' = \gamma(A_4 - i\beta A_1)$ using equ ⑨

$= \gamma [\Box^2 A_4 - i\beta \Box^2 A_1]$

$= \gamma [-\mu_0 J_4 - i\beta(-\mu_0 J_1)] \quad \because \Box^2 A_\mu = -\mu_0 J_\mu$ using equ ⑪

$= -\mu_0 [\gamma(J_4 - i\beta J_1)]$

$$\Box'^2 A_4' = -\mu_0 J_4' \text{ ——— } 14(d) \quad \because J_4' = \gamma(J_4 - i\beta J_1) \text{ using equ (10)}$$

Equns 14ⓐ, 14ⓑ, 14ⓒ and 14ⓓ can be jointly written in the form of single equation as

$$\Box'^2 A_\mu' = -\mu_0 J_\mu' \text{ ——— } (15)$$

Further we have

$$A_\mu' = a_{\mu\nu} A_\nu \quad \text{From equ ⑦}$$

$$\Rightarrow A_\mu' = \frac{\partial x_\mu'}{\partial x_\nu} \cdot A_\nu \quad \because x_\mu' = a_{\mu\nu} \cdot x_\nu \Rightarrow \frac{\partial x_\mu'}{\partial x_\nu} = a_{\mu\nu}$$

$$\Rightarrow \frac{\partial A_\mu'}{\partial x_\mu'} = \frac{\partial}{\partial x_\mu'} \cdot \frac{\partial x_\mu'}{\partial x_\nu} \cdot A_\nu$$

$$\Rightarrow \frac{\partial A_\mu'}{\partial x_\mu'} = \frac{\partial A_\nu}{\partial x_\nu}$$

But from Lorentz condition (12) $\frac{\partial A_\mu}{\partial x_\mu} = 0$

so $\frac{\partial A'_\mu}{\partial x'_\mu} = 0$ ——— (16)

Thus equations (15) and (16) are same as equation (13).

Therefore equations (11) and (12) retain the same form when expressed in terms of primed quantities.

So Maxwell's field equations are invariant under Lorentz transformation.

Ques:- To prove that Maxwell's field equations are invariant (or covariant) under Lorentz transformation.